Anomalous Transport from Fluid/Gravity Correspondence

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Yanyan Bu, M.L., Amir Sharon, 1608.08595 (JHEP), 1609.09054

Yanyan Bu and M.L., 1406.7222 (PRD), 1409.3095 (JHEP), 1502.08044 (JHEP)

Yanyan Bu, M.L., Amir Sharon, 1504.01370 (JHEP), 1511.08789 (JHEP)

Warm up: Electric Current

$$\vec{\mathbf{J}} = -\mathcal{D}^0 \vec{\nabla} \rho + \sigma_{\mathbf{e}}^0 \vec{\mathcal{E}}\,; \qquad \vec{\mathbf{J}}(\omega, \vec{\mathbf{q}}) = -\mathcal{D}^0 \mathbf{i} \vec{\mathbf{q}}\, \rho(\omega, \vec{\mathbf{q}}) + \sigma_{\mathbf{e}}^0 \vec{\mathcal{E}}(\omega, \vec{\mathbf{q}})$$

$\mathcal{D}^0$ **is a diffusion constant;** σ_{e}^0 **is a DC conductivity.**

Continuity equation (charge conservation): $\vec{\nabla} \vec{\mathbf{J}} + \dot{\rho} = \mathbf{0}$

Linear response theory:

$$\vec{\mathbf{J}}(\omega, \vec{\mathbf{q}}) \;=\; \sigma_{\mathbf{e}}^{\mathbf{AC}}(\omega)\, \vec{\mathcal{E}}(\omega, \vec{\mathbf{q}})\,; \qquad \sigma_{\mathbf{e}}^{\mathbf{AC}} \;=\; \frac{\mathbf{i}\omega\, \sigma_{\mathbf{e}}^{\mathbf{0}}}{\mathbf{i}\,\omega \;-\; \mathcal{D}^{\mathbf{0}}\, \mathbf{q}^{\mathbf{2}}}$$

AC conductivity. Valid in hydrodynamic regime of small ω, q**.**

Any generalisations to finite ω **and** q**?**

The most general (linear) constitutive relation for e/m current including both longitudinal and transverse responses

$$\vec{\mathbf{J}}(\omega, \vec{\mathbf{q}}) = -\mathcal{D}\left(\omega, \mathbf{q}^2\right) \mathbf{i}\vec{\mathbf{q}}\, \rho(\omega, \vec{\mathbf{q}}) + \sigma_{\mathbf{e}}\left(\omega, \mathbf{q}^2\right) \vec{\mathcal{E}}(\omega, \vec{\mathbf{q}}) + \sigma_{\mathbf{m}}\left(\omega, \mathbf{q}^2\right) \mathbf{i}\vec{\mathbf{q}} \times \vec{\mathcal{B}}\,(\omega, \vec{\mathbf{q}})\,.$$

$$\vec{\mathbf{J}} = -\mathcal{D}(\partial^{\mathrm{t}}, \nabla^{\mathbf{2}})\vec{\nabla}\, \rho + \sigma_{\mathbf{e}}\left(\partial^{\mathrm{t}}, \nabla^{\mathbf{2}}\right) \vec{\mathcal{E}} + \sigma_{\mathbf{m}}\left(\partial^{\mathrm{t}}, \nabla^{\mathbf{2}}\right) \vec{\nabla} \times \vec{\mathcal{B}}.$$

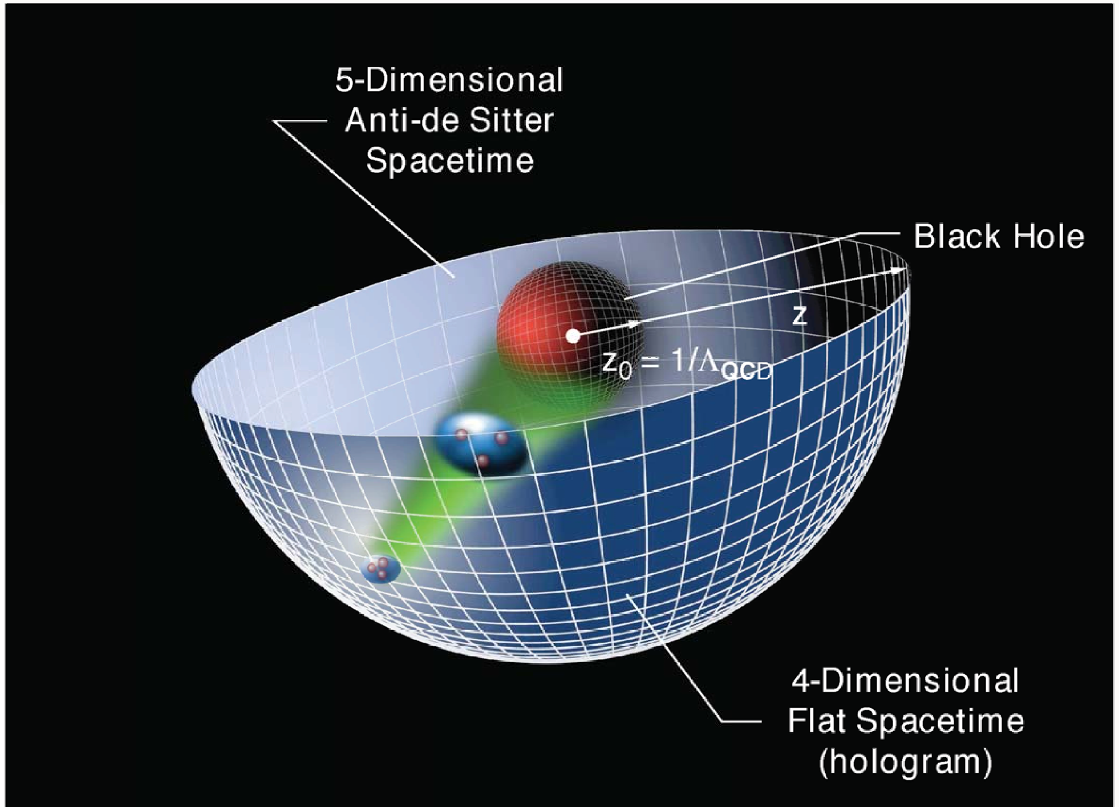
This is an "off-shell" constitutive relation.

$\mathcal{D}$, σ_e, and σ_m are momenta-dependent transport coefficient functions (TCF).

can be uniquely determined in holographic models to be discussed next

AdS/CFT

AdS/QCD



Changes in length scale mapped to evolution in the 5th dimension z

Maxwell field in Schwarzschild-AdS_5 geometry (probe approximation)

$$\mathbf{S} = - \int \mathbf{d^5x} \sqrt{-\mathbf{g}}\, \frac{\mathbf{1}}{\mathbf{4}}\, \mathbf{e^2}\, (\mathbf{F^V})_{\mathbf{MN}} (\mathbf{F^V})^{\mathbf{MN}} + \mathbf{S_{c.t.}}$$

Maxwell equations

$$\mathbf{EQ^N} := \nabla_{\mathbf{M}} \mathbf{F^{MN}} = \mathbf{0}$$

Schwarzschild-AdS_5 geometry (ingoing Eddington-Finkelstein coordinates)

$$\mathbf{ds^2} = \mathbf{g_{MN} dx^M dx^N} = \mathbf{2dtdr} - \mathbf{r^2 f(r) dt^2} + \mathbf{r^2 \delta_{ij} dx^i dx^j},$$

$\mathbf{f(r) = 1 - 1/r^4}$. **The horizon is at $\mathrm{r} = 1$, the Hawking temperature is $\pi\mathrm{T} = 1$.**

Near the conformal boundary $r = \infty$ the solution is expandable in a series ($A_r = 0$)

$$\mathbf{A}_\mu(\mathbf{r}, \mathbf{x}_\alpha) = \mathbf{A}_\mu^{(\mathbf{0})}(\mathbf{x}_\alpha) + \frac{\mathbf{A}_\mu^{(\mathbf{1})}(\mathbf{x}_\alpha)}{\mathbf{r}} + \frac{\mathbf{A}_\mu^{(\mathbf{2})}(\mathbf{x}_\alpha)}{\mathbf{r^2}} + \frac{\mathbf{B}_\mu^{(\mathbf{2})}(\mathbf{x}_\alpha)}{\mathbf{r^2}} \log \mathbf{r}^{-\mathbf{2}} + \mathcal{O}\left(\frac{\log \mathbf{r}^{-\mathbf{2}}}{\mathbf{r^3}}\right),$$

The boundary current (using the holographic dictionary)

$$\mathbf{J}^\mu = -\eta^{\mu\nu} \left(\mathbf{2A}_\nu^{(\mathbf{2})} + \mathbf{2B}_\nu^{(\mathbf{2})} + \eta^{\sigma \mathbf{t}} \partial_\sigma \mathbf{F}_{\mathbf{t}\nu}^{(\mathbf{0})} \right).$$

4 dynamical eqns $\mathbf{EQ}^{\mu} = \mathbf{0} \rightarrow$ transport, $\mathbf{EQ}^{\mathrm{r}} = \mathbf{0} \rightarrow$ current conservations.

$\mathbf{EQ}^{\mu} = \mathbf{0}$ admit the most general static homogeneous solutions

$$\mathbf{A}_{\mu} = \mathbf{A}_{\mu}^{(0)} + \frac{\rho}{\mathbf{2r^2}}\delta_{\mu \mathrm{t}}, \qquad \mathbf{A}_{\mu}^{(0)} = \mathbf{const}, \quad \rho = \mathbf{const}$$

The boundary theory is a static uniformly charged plasma with no external fields

$$\mathbf{J}^{\mathrm{t}} = \rho, \qquad \mathbf{J}^{\mathrm{i}} = \mathbf{0}$$

Next, following the spirit of

S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, (2008)

$$\mathbf{A}_{\mu}^{(0)} \rightarrow \mathbf{A}_{\mu}^{(0)}(\mathbf{x}_{\alpha}), \qquad \rho \rightarrow \rho(\mathbf{x}_{\alpha}).$$

The solution has to be amended:

$$\mathbf{A}_{\mu}(\mathbf{r}, \mathbf{x}_{\alpha}) = \mathbf{A}_{\mu}^{(0)}(\mathbf{x}_{\alpha}) + \frac{\rho(\mathbf{x}_{\alpha})}{\mathbf{2r^2}}\delta_{\mu \mathrm{t}} + \mathbf{a}_{\mu}(\mathbf{r}, \mathbf{x}_{\alpha})$$

Solve for a (bulk-to-boundary propagator)

Different from approaches based on two-point correlators, which assume on-shellness

$U(1)$ vector current: Diffusion and Conductivity

$$\vec{\mathbf{J}}(\omega,\vec{\mathbf{q}}) = -\mathcal{D}\left(\omega,\mathbf{q}^2\right)\mathbf{i}\vec{\mathbf{q}}\,\rho(\omega,\vec{\mathbf{q}}) + \sigma_{\mathbf{e}}\left(\omega,\mathbf{q}^2\right)\vec{\mathbf{E}}(\omega,\vec{\mathbf{q}}) + \sigma_{\mathbf{m}}\left(\omega,\mathbf{q}^2\right)\mathbf{i}\vec{\mathbf{q}}\times\vec{\mathbf{B}}(\omega,\vec{\mathbf{q}})\,.$$

$$\mathcal{D} = \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{8}}\pi\mathbf{i}\omega + \frac{\mathbf{1}}{\mathbf{48}}\left[-\pi^{\mathbf{2}}\omega^{\mathbf{2}} + \mathbf{q}^{\mathbf{2}}\left(\mathbf{6}\log\mathbf{2} - \mathbf{3}\pi\right)\right] + \cdots,$$

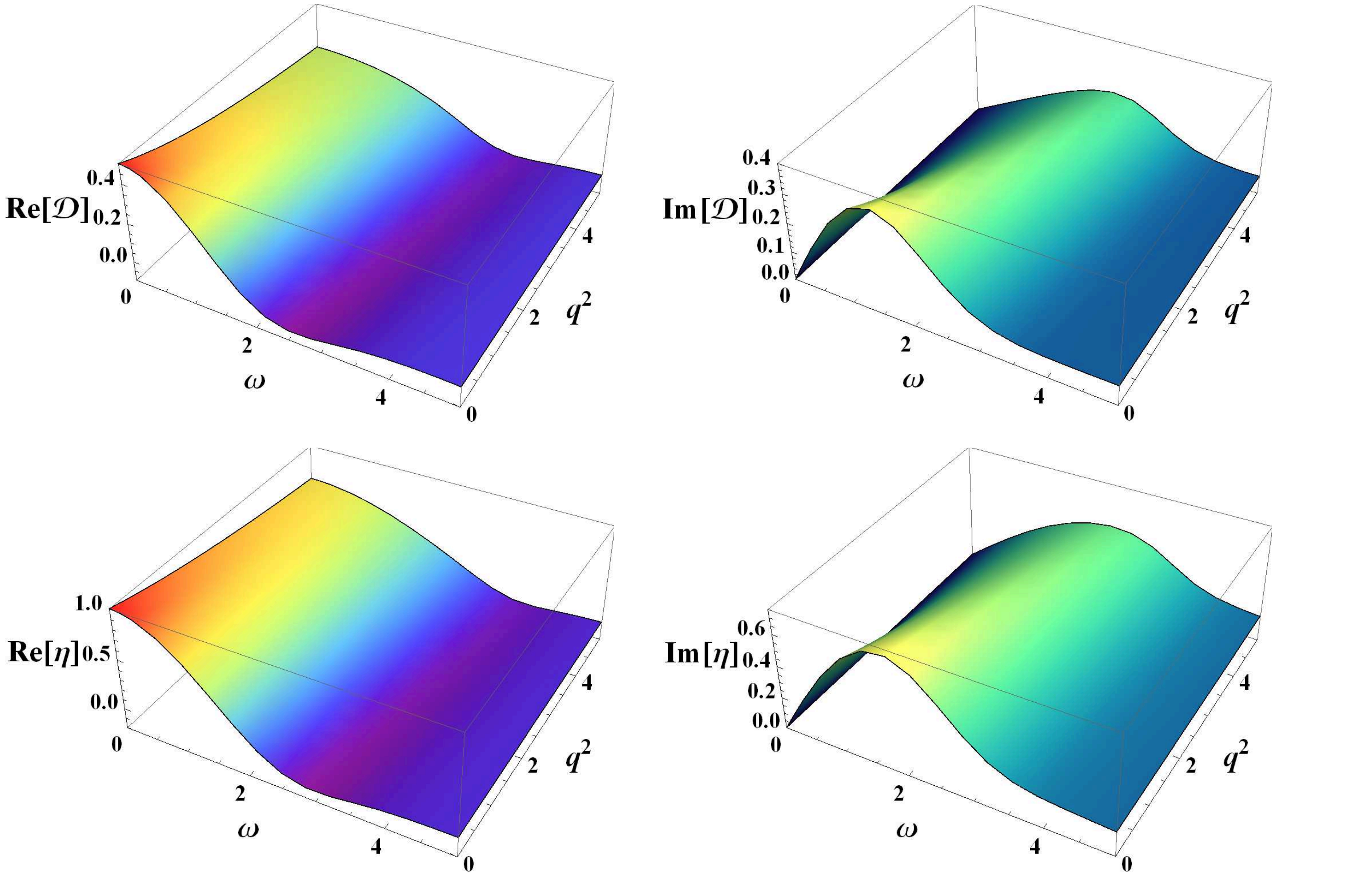
$$\sigma_{\mathbf{e}} = \mathbf{1} + \frac{\log\mathbf{2}}{\mathbf{2}}\mathbf{i}\omega + \frac{\mathbf{1}}{\mathbf{24}}\left[\pi^{\mathbf{2}}\omega^{\mathbf{2}} - \mathbf{q}^{\mathbf{2}}\left(\mathbf{3}\pi + \mathbf{6}\log\mathbf{2}\right)\right] + \cdots,$$

$$\sigma_{\mathbf{m}} = \mathbf{0} + \frac{\mathbf{1}}{\mathbf{16}}\mathbf{i}\omega\left(\mathbf{2}\pi - \pi^{\mathbf{2}} + \mathbf{4}\log\mathbf{2}\right) + \cdots.$$

$\sigma_{\mathbf{m}}^{\mathbf{0}} > \mathbf{0}$ in a pure QED plasma with one Dirac fermion at one loop level
B. B. Brandt, A. Francis, and H. B. Meyer, (2014)

$\sigma_{\mathbf{m}}^{\mathbf{0}} = \mathbf{0}$ based on Boltzmann equations J. Hong and D. Teaney, (2010)

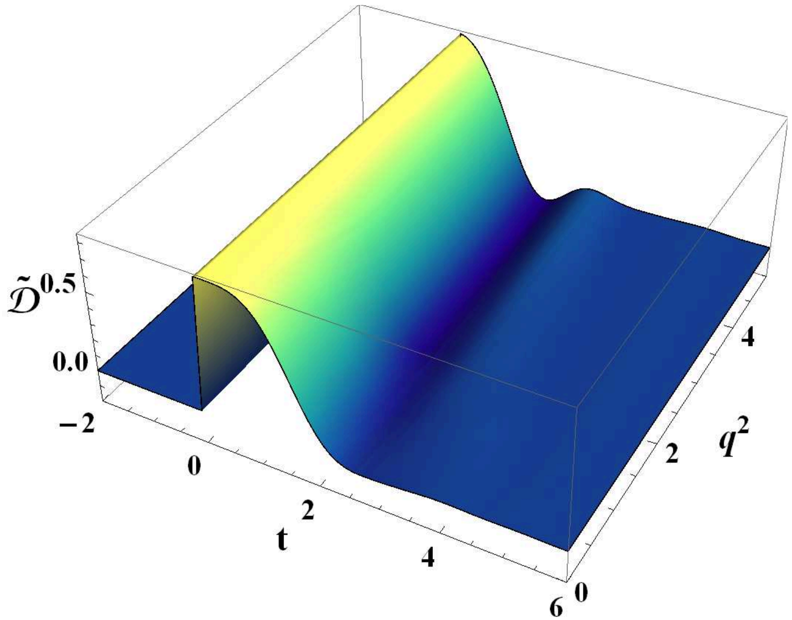
Maxwell is linear (exact), no Lorentz force

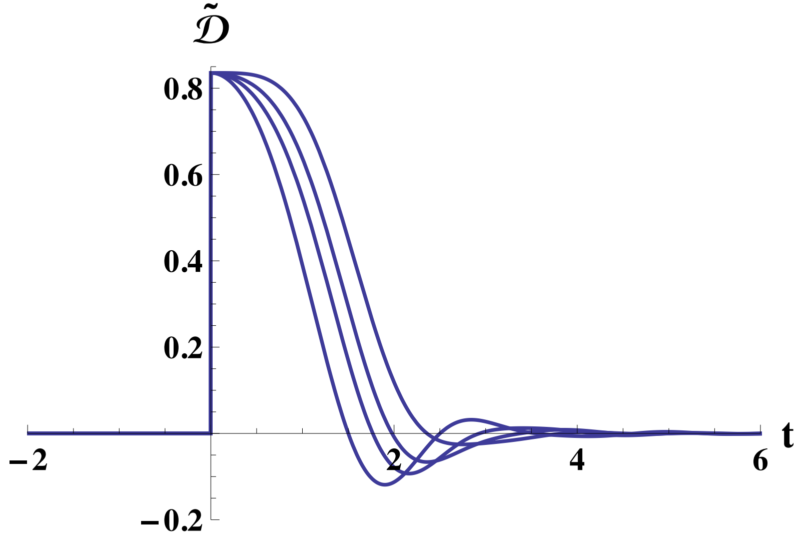


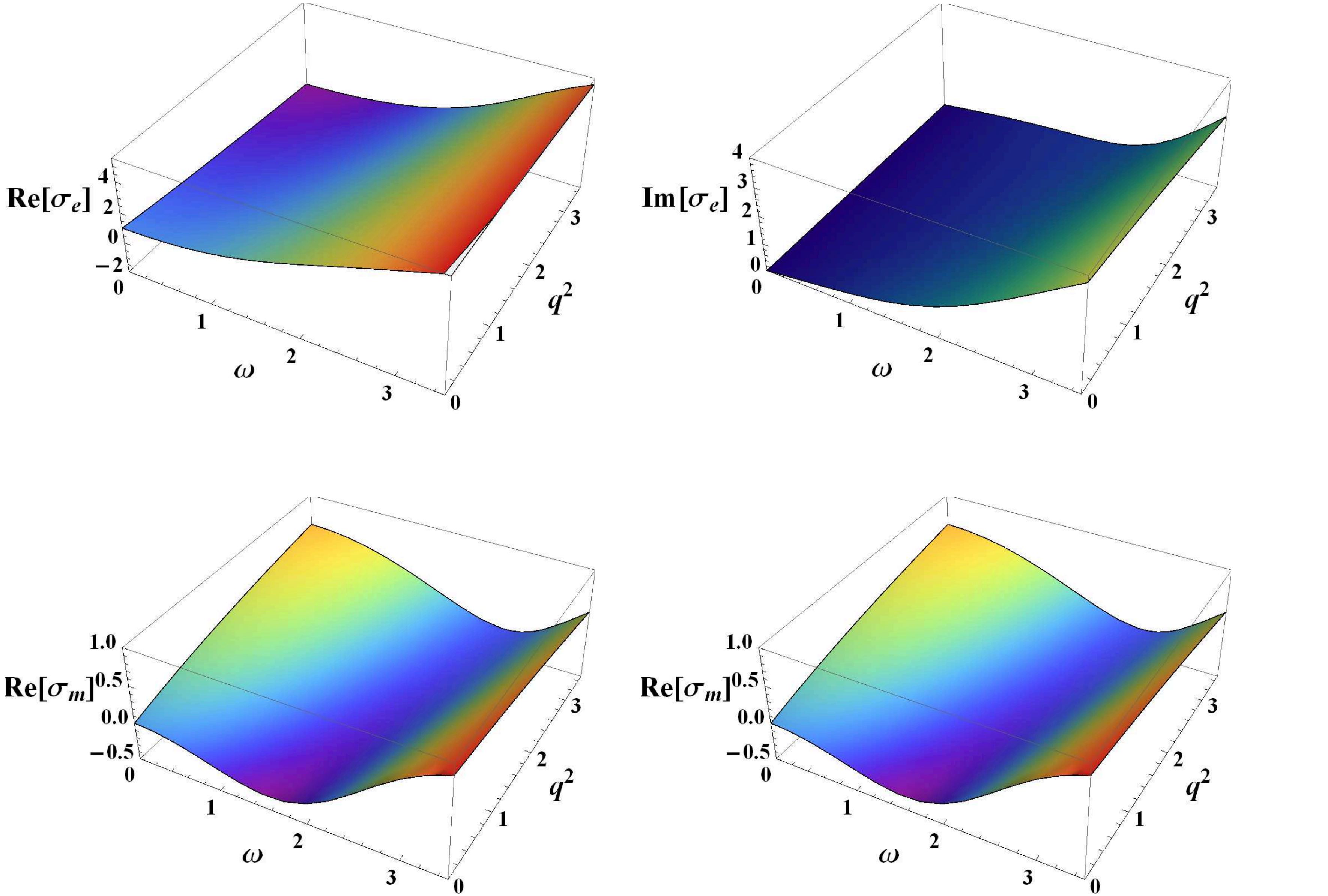
Re[𝒟]
0.4
0.2
0.0
0
2
4
ω
q²
Im[𝒟]
0.4
0.3
0.2
0.1
0.0
Re[η]
1.0
0.5
0.0
Im[η]
0.6
0.4
0.2
0.0

Memory Function / Causality

$$\widetilde{\mathcal{D}}\left(\mathbf{t},\mathbf{q^2}\right)=\frac{\mathbf{1}}{\sqrt{\mathbf{2}\pi}}\int_{-\infty}^{\infty}\mathcal{D}\left(\omega,\mathbf{q^2}\right)\mathbf{e}^{-\mathbf{i}\omega\mathbf{t}}\mathbf{d}\omega$$







Re[σ_e]
ω
q^2
Im[σ_e]
ω
q^2
Re[σ_m]
ω
q^2
Re[σ_m]
ω
q^2

$\mathbf{U_V(1) \times U_A(1)}$: Anomaly-induced transport

5d Lagrangian:

$$\mathcal{L} = -\frac{1}{4}\mathbf{e}^2(\mathbf{F}^V)_{MN}(\mathbf{F}^V)^{MN} - \frac{1}{4}\mathbf{e}'^2(\mathbf{F}^a)_{MN}(\mathbf{F}^a)^{MN}$$
$$+\frac{\kappa\,\epsilon^{MNPQR}}{2\sqrt{-g}}\left[3\mathbf{e}^2\mathbf{e}'\mathbf{A}_M(\mathbf{F}^V)_{NP}(\mathbf{F}^V)_{QR} + \mathbf{e}'^3\mathbf{A}_M(\mathbf{F}^a)_{NP}(\mathbf{F}^a)_{QR}\right].$$

Boundary currents:

$$\partial_\mu \mathbf{J}^\mu = \mathbf{0}, \qquad \partial_\mu \mathbf{J}_5^\mu = 4\kappa\left(3\vec{\mathcal{E}}\cdot\vec{\mathcal{B}} + \vec{\mathcal{E}}^a\cdot\vec{\mathcal{B}}^a\right)$$

external e/m ($\vec{\mathcal{E}},\ \vec{\mathcal{B}}$) and axial ($\vec{\mathcal{E}}^a,\ \vec{\mathcal{B}}^a$) fields.

CS introduces non-linearity in EQ

I) Linear transport

$$\rho(\mathbf{x}_\alpha) = \bar\rho + \epsilon\delta\rho(\mathbf{x}_\alpha), \qquad \rho_\mathbf{5}(\mathbf{x}_\alpha) = \bar\rho_\mathbf{5} + \epsilon\delta\rho_\mathbf{5}(\mathbf{x}_\alpha),$$

$$\mu(\mathbf{x}_\alpha) = \bar\mu + \epsilon\delta\mu(\mathbf{x}_\alpha), \qquad \mu_\mathbf{5}(\mathbf{x}_\alpha) = \bar\mu_\mathbf{5} + \epsilon\delta\mu_\mathbf{5}(\mathbf{x}_\alpha), \qquad \bar\mu = \bar\rho/\mathbf{2}, \qquad \bar\mu_\mathbf{5} = \bar\rho_\mathbf{5}/\mathbf{2}$$

$$\mathcal{E}_\mathrm{i}(\mathbf{x}_\alpha) \to \epsilon\mathcal{E}_\mathrm{i}(\mathbf{x}_\alpha), \qquad \mathcal{B}_\mathrm{i}(\mathbf{x}_\alpha) \to \epsilon\mathcal{B}_\mathrm{i}(\mathbf{x}_\alpha), \qquad \mathcal{E}^\mathbf{a}_\mathrm{i}(\mathbf{x}_\alpha) \to \epsilon\mathcal{E}^\mathbf{a}_\mathrm{i}(\mathbf{x}_\alpha), \qquad \mathcal{B}_\mathrm{i}(\mathbf{x}_\alpha) \to \epsilon\mathcal{B}^\mathbf{a}_\mathrm{i}(\mathbf{x}_\alpha).$$

$$\mathbf{J}^\mathrm{t} = \rho, \qquad \vec{\mathbf{J}} = -\mathcal{D}\vec\nabla\rho + \sigma_\mathbf{e}\vec{\mathcal{E}} + \sigma_\mathbf{m}\vec\nabla\times\vec{\mathcal{B}} + \sigma_\chi\vec{\mathcal{B}} + \sigma_\mathbf{a}\vec\nabla\times\vec{\mathbf{B}}^\mathbf{a} + \sigma_\kappa\vec{\mathcal{B}}^\mathbf{a}$$

$$\mathbf{J}^\mathrm{t}_\mathbf{5} = \rho_\mathbf{5}, \qquad \mathbf{J}^\mathrm{i}_\mathbf{5} = -\mathcal{D}\vec\nabla\rho_\mathbf{5} + \sigma_\mathbf{e}\vec{\mathcal{E}}^\mathbf{a} + \sigma_\mathbf{m}\vec\nabla\times\vec{\mathcal{B}}^\mathbf{a} + \sigma_\chi\vec{\mathcal{B}}^\mathbf{a} + \sigma_\mathbf{a}\vec\nabla\times\vec{\mathcal{B}} + \sigma_\kappa\vec{\mathcal{B}}. \qquad \partial_\mu\mathbf{J}^\mu_\mathbf{5} = \mathbf{0}$$

σ_χ – **CME**; D. E. Kharzeev and H. J. Warringa, (2009)

σ_κ – **CSE**; D. T. Son and A. R. Zhitnitsky, (2004) ; M. A. Metlitski and A. R. Zhitnitsky, (2005)

$$\sigma_{\mathbf{m}} = \mathbf{72}\kappa^{\mathbf{2}}\left(\bar{\mu}^{\mathbf{2}} + \bar{\mu}_{\mathbf{5}}^{\mathbf{2}}\right)(\mathbf{2}\log\mathbf{2} - \mathbf{1}) + \mathbf{i}\omega\left[\frac{\mathbf{1}}{\mathbf{16}}(\mathbf{2}\pi - \pi^{\mathbf{2}} + \mathbf{4}\log\mathbf{2}) + \mathcal{O}\left(\bar{\mu}^{\mathbf{2}} + \bar{\mu}_{\mathbf{5}}^{\mathbf{2}}\right)\right] + \cdots,$$

$\sigma_{\mathbf{m}}[\mathbf{q} = \mathbf{0}] - \sigma_{\mathbf{m}}[\mathbf{q} = \mathbf{0},\ \bar{\mu} = \bar{\mu}_{\mathbf{5}} = \mathbf{0}]$ **is linear in** $\kappa^2\,(\bar{\mu}^2 + \bar{\mu}_5^2)$

$$\sigma_\chi = \mathbf{12}\kappa\bar{\mu}_{\mathbf{5}}\left\{\mathbf{1} + \mathbf{i}\omega\log\mathbf{2} - \frac{\mathbf{1}}{\mathbf{4}}\omega^{\mathbf{2}}\log^{\mathbf{2}}\mathbf{2} - \frac{\mathbf{q}^{\mathbf{2}}}{\mathbf{24}}\left[\pi^{\mathbf{2}} - \mathbf{1728}\kappa^{\mathbf{2}}\left(\bar{\mu}_{\mathbf{5}}^{\mathbf{2}} + \mathbf{3}\bar{\mu}^{\mathbf{2}}\right)(\log\mathbf{2} - \mathbf{1})^{\mathbf{2}}\right]\right\} + \cdots,$$

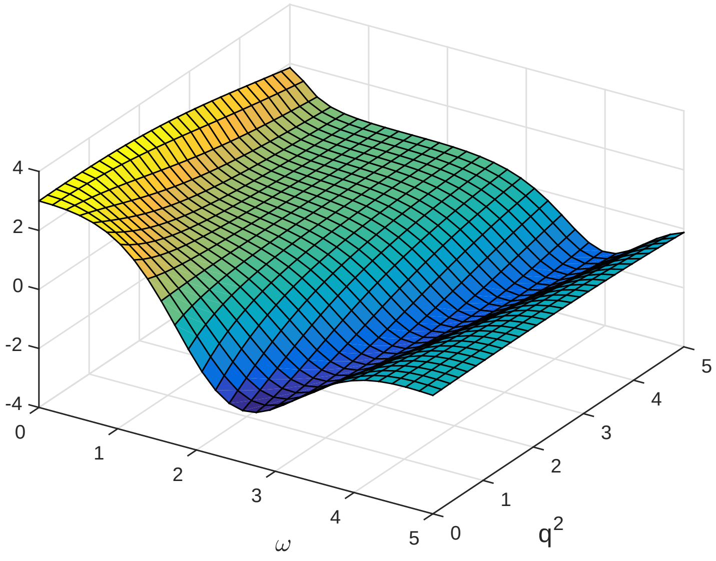
$\sigma_\chi^{\mathbf{0}}$ **A. Gynther, K. Landsteiner, F. Pena-Benitez, and A. Rebhan, (2011)**

$\sigma_\chi[\mathbf{q} = 0]$ **is linear in** $\kappa\,\mu_5$ **and independent of** μ

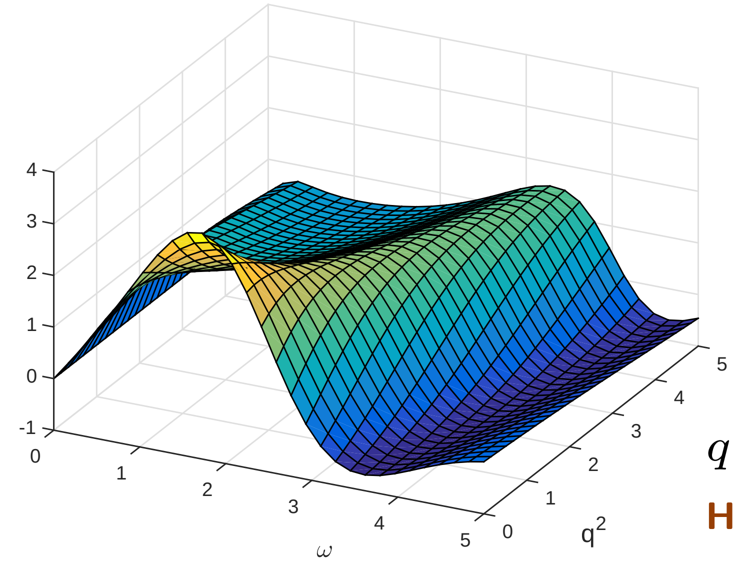
$$\sigma_\kappa[\mu,\ \mu_5] = \sigma_\chi[\mu_5,\ \mu] \qquad\qquad \sigma_a = 144\kappa^2\bar{\mu}\bar{\mu}_5\,(2\ln 2 - 1) + \cdots,$$

Plus tons of plots for arbitrary $\omega,\ q$ **and** $\mu,\ \mu_5$

$\mathrm{Re}(\sigma_\chi),\ \kappa\bar{\mu}_5 = 0.25,\ \kappa\bar{\mu} = 0$



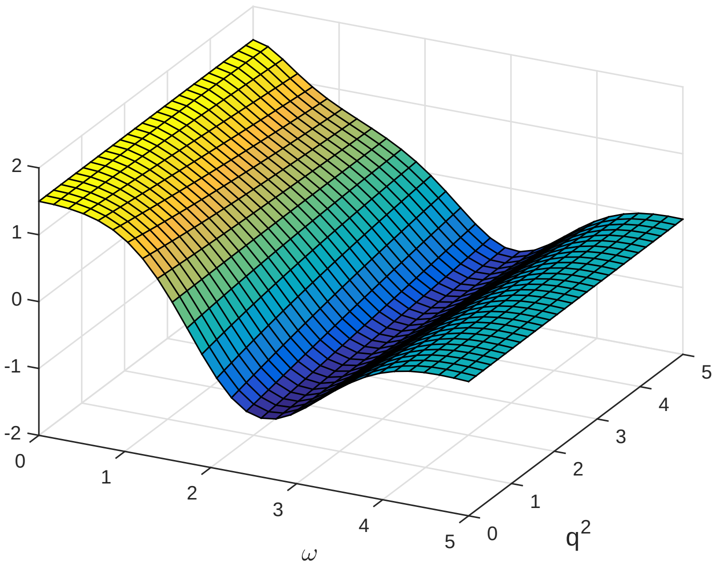
$\mathrm{Im}(\sigma_\chi),\ \kappa\bar{\mu}_5 = 0.25,\ \kappa\bar{\mu} = 0$



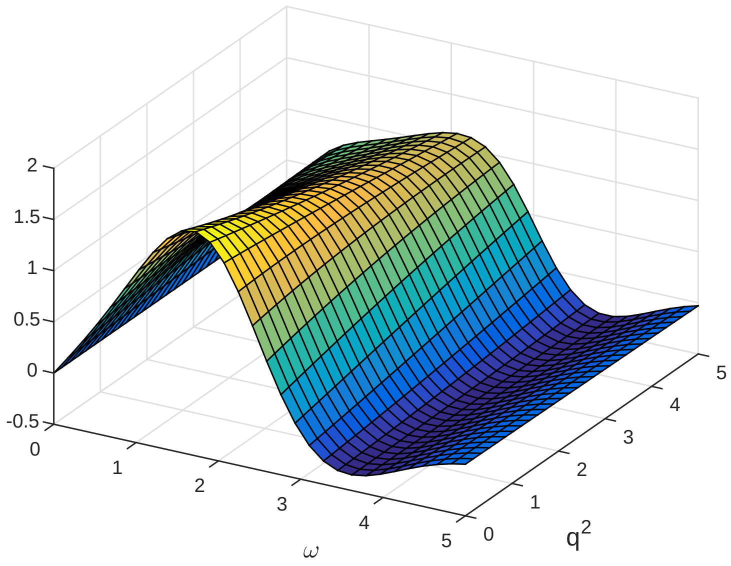
$q = 0$

H.-U. Yee, (2009)

$\mathrm{Re}(\sigma_\chi),\ \kappa\bar{\mu}_5 = 0.125,\ \kappa\bar{\mu} = 0.125$



$\mathrm{Im}(\sigma_\chi),\ \kappa\bar{\mu}_5 = 0.125,\ \kappa\bar{\mu} = 0.125$



II) Non-linear corrections induced by the magnetic field

$$\vec{\mathcal{B}} = \vec{\mathcal{B}}(\vec{\mathbf{x}}) \neq \vec{\mathcal{B}}(\mathbf{t})$$

$$\mathcal{D}_0 = \frac{\mathbf{1}}{\mathbf{2}} - \mathbf{18}(\mathbf{2}\log\mathbf{2} - \mathbf{1})\ \kappa^2\,\mathcal{B}^2$$

The correction is negative! Violates low bounds P. Kovtun and A. Ritz, (2008)

$$\omega = \left[\mp\mathbf{1} + \mathbf{36}\,(\mathbf{1} - \mathbf{2}\log\mathbf{2})\,\kappa^2\mathcal{B}^2\right]\mathbf{6}\kappa\vec{\mathbf{q}}\cdot\vec{\mathcal{B}} - \left[\frac{\mathbf{1}}{\mathbf{2}} + \mathbf{18}\,(\mathbf{1} - \mathbf{2}\log\mathbf{2})\,\kappa^2\mathcal{B}^2\right]\mathbf{i}\mathbf{q}^2 - \frac{\mathbf{i}}{\mathbf{8}}\mathbf{q}^4\log\mathbf{2} + \cdots.$$

The first term represents the chiral magnetic wave (CMW) D. E. Kharzeev and H.-U. Yee, (2011)

We see nonlinear in $\mathcal{B}$ corrections to both the speed of CMW and its decay rate.
Also the degeneracy in the speed of the positive and negative modes is removed.

III) Non-linear transport induced by constant external e/m fields

The external fields are assumed to have constant background values $\vec{\mathbf{E}}, \vec{\mathbf{B}}, \vec{\mathbf{E}}^a, \vec{\mathbf{B}}^a$

$$\vec{\mathcal{E}}(\mathbf{x}_\alpha) = \vec{\mathbf{E}} + \epsilon\delta\vec{\mathbf{E}}(\mathbf{x}_\alpha), \qquad \vec{\mathcal{B}}(\mathbf{x}_\alpha) = \vec{\mathbf{B}} + \epsilon\delta\vec{\mathbf{B}}(\mathbf{x}_\alpha),$$

$$\vec{\mathcal{E}}^{\mathbf{a}}(\mathbf{x}_\alpha) = \vec{\mathbf{E}}^{\mathbf{a}} + \epsilon\delta\vec{\mathbf{E}}^{\mathbf{a}}(\mathbf{x}_\alpha), \qquad \vec{\mathcal{B}}^{\mathbf{a}}(\mathbf{x}_\alpha) = \vec{\mathbf{B}}^{\mathbf{a}} + \epsilon\delta\vec{\mathbf{E}}^{\mathbf{a}}(\mathbf{x}_\alpha).$$

- Constant background $\vec{\mathbf{B}} \neq 0$, $\quad \vec{E} = \vec{B}^a = \vec{E}^a = 0$

$$\mathbf{J}^{\mathrm{t}}_{(\mathbf{0})} = \bar{\rho}, \qquad \vec{\mathbf{J}}_{(\mathbf{0})} = \mathbf{12}\kappa\mu_{\mathbf{5}}(\mathbf{B})\,\vec{\mathbf{B}} \qquad \mathbf{J}^{\mathrm{t}}_{\mathbf{5}(\mathbf{0})} = \bar{\rho}_{\mathbf{5}}, \qquad \vec{\mathbf{J}}_{\mathbf{5}(\mathbf{0})} = \mathbf{12}\kappa\mu(\mathbf{B})\,\vec{\mathbf{B}}$$

Index $_{(0)}$ means zeroth order in gradient expansion $(\epsilon = 0)$

CME is exact to all orders in $\vec{\mathrm{B}}$. D. E. Kharzeev and H.-U. Yee (2011); A. V. Sadofyev and M. V. Isachenkov (2011), U. Gursoy and A. Jansen (2014); U. Gursoy and J. Tarrio (2015)
but there are gradient and $\vec{\mathrm{E}}$-field corrections

- Constant background e/m fields $\vec{\mathbf{B}}$ & $\vec{\mathbf{E}}$, weak field expansion, $\vec{\mathcal{B}}^a = \vec{\mathcal{E}}^a = 0$

$$\vec{\mathbf{J}}_0 = \vec{\mathbf{E}} + \mathbf{12}\kappa\mu_5(\mathbf{B},\mathbf{E})\vec{\mathbf{B}} + \mathbf{72}\log\mathbf{2}\,\kappa^2\mu(\mathbf{B})\vec{\mathbf{E}}\times\vec{\mathbf{B}} - \mathbf{36}\pi^2\kappa^3\mu_5(\mathbf{B},\mathbf{E})\left(\vec{\mathbf{B}}\times\vec{\mathbf{E}}\right)\times\vec{\mathbf{E}} + \cdots,$$

$$\vec{\mathbf{J}}_{5(0)} = \mathbf{12}\kappa\mu(\mathbf{B})\vec{\mathbf{B}} + \mathbf{72}\log\mathbf{2}\,\kappa^2\mu_5(\mathbf{B},\mathbf{E})\vec{\mathbf{B}}\times\vec{\mathbf{E}} - \mathbf{36}\pi^2\kappa^3\mu(\mathbf{B})\left(\vec{\mathbf{B}}\times\vec{\mathbf{E}}\right)\times\vec{\mathbf{E}} + \cdots,$$

Field-dependent chemical potentials

$$\mu(\mathbf{B}) = \frac{\mathbf{1}}{\mathbf{2}}\bar{\rho} + \mathbf{18}\,(\mathbf{1} - \mathbf{2}\log\mathbf{2})\,\kappa^2\bar{\rho}\mathbf{B}^2 + \cdots,$$

$$\mu_5(\mathbf{B},\mathbf{E}) = \frac{\mathbf{1}}{\mathbf{2}}\bar{\rho}_5 + \mathbf{18}\,(\mathbf{1} - \mathbf{2}\log\mathbf{2})\,\kappa^2\bar{\rho}_5\mathbf{B}^2 + \frac{\mathbf{1}}{\mathbf{8}}\,(-\pi + \mathbf{2}\log\mathbf{2})\,\mathbf{12}\kappa\vec{\mathbf{B}}\cdot\vec{\mathbf{E}} + \cdots.$$

- μ_5 is induced even in totally neutral plasmas $\bar{\rho} = \bar{\rho}_5 = 0$, via $(\vec{E}\cdot\vec{B})$.

- $\mathrm{B}^2\vec{\mathbf{B}}$ & $(\vec{\mathbf{B}}\vec{\mathbf{E}})\vec{\mathbf{B}}$ terms are the first nonlinear effects in CME
important for discussions of strong magnetic fields

$$\vec{\mathbf{J}}_0 = \vec{\mathbf{E}} + 12\kappa\mu_5(\mathbf{B},\mathbf{E})\vec{\mathbf{B}} + 72\log 2\,\kappa^2\mu(\mathbf{B})\vec{\mathbf{E}}\times\vec{\mathbf{B}} - 36\pi^2\kappa^3\mu_5(\mathbf{B},\mathbf{E})\left(\vec{\mathbf{B}}\times\vec{\mathbf{E}}\right)\times\vec{\mathbf{E}} + \cdots,$$

$$\vec{\mathbf{J}}_{5(0)} = 12\kappa\mu(\mathbf{B})\vec{\mathbf{B}} + 72\log 2\,\kappa^2\mu_5(\mathbf{B},\mathbf{E})\vec{\mathbf{B}}\times\vec{\mathbf{E}} - 36\pi^2\kappa^3\mu(\mathbf{B})\left(\vec{\mathbf{B}}\times\vec{\mathbf{E}}\right)\times\vec{\mathbf{E}} + \cdots,$$

- $\left(\vec{\mathbf{B}}\times\vec{\mathbf{E}}\right)$ term leads to anomaly induced Hall current

  (chiral Hall effect S. Pu, S.-Y. Wu, and D.-L. Yang, (2015))

- $\vec{\mathbf{J}}_0 \sim (\vec{\mathbf{B}}\times\vec{\mathbf{E}})\times\vec{\mathbf{E}} = -\mathbf{E}^2\vec{\mathbf{B}}$ (CME) $+ (\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})\vec{\mathbf{E}}$ (CEE)

(χKT E. V. Gorbar, I. A. Shovkovy, S. Vilchinskii, I. Rudenok, A. Boyarsky, and O. Ruchayskiy, (2016))

CEE Chiral Electric Effect Y. Neiman and Y. Oz, (2011)

- $\vec{\mathbf{J}}_{5(0)} \sim (\vec{\mathbf{B}}\times\vec{\mathbf{E}})\times\vec{\mathbf{E}} = -\mathbf{E}^2\vec{\mathbf{B}}$ (CSE) $+ (\vec{\mathbf{E}}\cdot\vec{\mathbf{B}})\vec{\mathbf{E}}$ (CESE)

CESE Chiral Electric Separation Effect X.-G. Huang and J. Liao, (2013), even when $\mu_5 = 0$

Towards all order non-linear constitutive relations

Linear in constant background times linear in inhomogeneous field perturbations

$$
\begin{gathered}
\delta\rho_5 \vec{\mathbf{B}}, \quad \delta\rho\, \vec{\mathbf{B}}^a, \quad \left(\vec{\nabla}\cdot\delta\vec{E}^a\right)\vec{\mathbf{B}}, \quad \left(\vec{\nabla}\cdot\delta\vec{E}\right)\vec{\mathbf{B}}^a, \quad \vec{\mathbf{E}}^a\times\delta\vec{E}, \quad \vec{\mathbf{E}}^a\times\vec{\nabla}\delta\rho, \\
\vec{\mathbf{E}}^a\times\left(\vec{\nabla}\times\delta\vec{B}\right), \quad \vec{\mathbf{E}}^a\times\delta\vec{B}, \quad \vec{\mathbf{E}}^a\times\left(\vec{\nabla}\times\delta\vec{B}^a\right), \quad \vec{\mathbf{E}}^a\times\delta\vec{B}^a, \\
\vec{\mathbf{E}}^a\times\vec{\nabla}\left(\vec{\nabla}\cdot\delta\vec{E}\right), \quad \left(\bar{\rho}\vec{\mathbf{B}}+\bar{\rho}_5\vec{\mathbf{B}}^a\right)\times\delta\vec{E}, \quad \left(\bar{\rho}\vec{\mathbf{B}}+\bar{\rho}_5\vec{\mathbf{B}}^a\right)\times\vec{\nabla}\delta\rho, \\
\left(\bar{\rho}\vec{\mathbf{B}}+\bar{\rho}_5\vec{\mathbf{B}}^a\right)\times\left(\vec{\nabla}\times\delta\vec{B}\right), \quad \left(\bar{\rho}\vec{\mathbf{B}}+\bar{\rho}_5\vec{\mathbf{B}}^a\right)\times\delta\vec{B}, \quad \left(\bar{\rho}\vec{\mathbf{B}}+\bar{\rho}_5\vec{\mathbf{B}}^a\right)\times\left(\vec{\nabla}\times\delta\vec{B}^a\right), \\
\left(\bar{\rho}\vec{\mathbf{B}}+\bar{\rho}_5\vec{\mathbf{B}}^a\right)\times\delta\vec{B}^a, \quad \left(\bar{\rho}\vec{\mathbf{B}}+\bar{\rho}_5\vec{\mathbf{B}}^a\right)\times\vec{\nabla}\left(\vec{\nabla}\cdot\delta\vec{E}\right), \quad \vec{\mathbf{E}}\times\delta\vec{E}^a, \quad \vec{\mathbf{E}}\times\vec{\nabla}\delta\rho_5, \\
\vec{\mathbf{E}}\times\left(\vec{\nabla}\times\delta\vec{B}^a\right), \quad \vec{\mathbf{E}}\times\delta\vec{B}^a, \quad \vec{\mathbf{E}}\times\left(\vec{\nabla}\times\delta\vec{B}\right), \quad \vec{\mathbf{E}}\times\delta\vec{B}, \quad \vec{\mathbf{E}}\times\vec{\nabla}\left(\vec{\nabla}\cdot\delta\vec{E}^a\right), \\
\left(\bar{\rho}_5\vec{\mathbf{B}}+\bar{\rho}\vec{\mathbf{B}}^a\right)\times\delta\vec{E}^a, \quad \left(\bar{\rho}_5\vec{\mathbf{B}}+\bar{\rho}\vec{\mathbf{B}}^a\right)\times\vec{\nabla}\delta\rho_5, \quad \left(\bar{\rho}_5\vec{\mathbf{B}}+\bar{\rho}\vec{\mathbf{B}}^a\right)\times\left(\vec{\nabla}\times\delta\vec{B}^a\right), \\
\left(\bar{\rho}_5\vec{\mathbf{B}}+\bar{\rho}\vec{\mathbf{B}}^a\right)\times\delta\vec{B}^a, \quad \left(\bar{\rho}_5\vec{\mathbf{B}}+\bar{\rho}\vec{\mathbf{B}}^a\right)\times\left(\vec{\nabla}\times\delta\vec{B}\right), \quad \left(\bar{\rho}_5\vec{\mathbf{B}}+\bar{\rho}\vec{\mathbf{B}}^a\right)\times\delta\vec{B}, \\
\left(\bar{\rho}_5\vec{\mathbf{B}}+\bar{\rho}\vec{\mathbf{B}}^a\right)\times\vec{\nabla}\left(\vec{\nabla}\cdot\delta\vec{E}^a\right).
\end{gathered}
$$

The first terms $\delta\rho_5\vec{\mathrm{B}}, \delta\rho\vec{\mathrm{B}}$ are responsible for the chiral magnetic wave (CMW)

IV) Constant magnetic and time-dependent electric fields

$$\vec{\mathcal{B}} = \vec{\mathbf{B}}, \qquad \vec{\mathcal{E}} = \vec{\mathcal{E}}(\mathbf{t}) \neq \vec{\mathcal{E}}(\vec{\mathbf{x}}), \qquad \vec{\mathcal{B}}^{\mathbf{a}} = \vec{\mathcal{E}}^{\mathbf{a}} = \mathbf{0}, \qquad \bar{\rho} = \bar{\rho}_5 = \mathbf{0}$$

$$\partial_\mu \mathbf{J}^\mu = \mathbf{0}, \qquad \partial_\mu \mathbf{J}_5^\mu = \mathbf{12}\kappa \vec{\mathcal{E}} \cdot \vec{\mathbf{B}} \longrightarrow \mu_5 \neq \mathbf{0}$$

- **Gradient expansion**

$$\vec{\mathbf{J}} = \mathbf{12}\kappa\mu_5\vec{\mathbf{B}} + \vec{\mathcal{E}} - \frac{\log \mathbf{2}}{\mathbf{2}}\partial_t\vec{\mathcal{E}} - \frac{\pi^2}{\mathbf{24}}\partial_t^2\vec{\mathcal{E}} - \left(\frac{\mathbf{3}}{\mathbf{2}}\pi + \mathbf{3}\log\mathbf{2}\right)\kappa\partial_t\rho_5\vec{\mathbf{B}} + \mathbf{9}\pi^2\kappa^3\rho_5\left(\vec{\mathbf{B}} \times \vec{\mathcal{E}}\right) \times \vec{\mathcal{E}} + \mathbf{12}\#_1\kappa\partial_t^2\rho_5\vec{\mathbf{B}} + \mathcal{O}\left(\partial^4\right),$$

$$\vec{\mathbf{J}}_5 = \mathbf{12}\kappa\mu\vec{\mathbf{B}} - \mathbf{36}\log\mathbf{2}\,\kappa^2\rho_5\vec{\mathbf{B}} \times \vec{\mathcal{E}} + \frac{\mathbf{3}}{\mathbf{2}}\left(\pi^2 + \mathbf{3}\pi\log\mathbf{2} + \mathbf{6}\log^2\mathbf{2}\right)\kappa^2\partial_t\rho_5\vec{\mathbf{B}} \times \vec{\mathcal{E}} - \frac{\mathbf{3}}{\mathbf{8}}\left(\mathbf{48}\mathcal{C} + \pi^2 - \mathbf{12}\pi\log\mathbf{2}\right)\kappa^2\rho_5\vec{\mathbf{B}} \times \partial_t\vec{\mathcal{E}} + \mathcal{O}\left(\partial^4\right),$$

where $\mathcal{C}$ is a Catalan constant and $\#_1$ is known numerically only $\#_1 \approx 0.362$.

$$\mu = \mathbf{0} + \mathcal{O}\left(\partial^3\right), \quad \mu_5 = \frac{\mathbf{1}}{\mathbf{2}}\rho_5 + \frac{\mathbf{3}}{\mathbf{2}}\left(\pi - \mathbf{2}\log\mathbf{2}\right)\kappa\vec{\mathcal{E}} \cdot \vec{\mathbf{B}} + \mathbf{18}\left(\mathbf{1} - \mathbf{2}\log\mathbf{2}\right)\kappa^2\rho_5\mathbf{B}^2 + \mathcal{O}\left(\partial^3\right).$$

- **Weak electric field expansion**

$$\rho_5 \sim \mathcal{O}(\epsilon), \qquad \vec{\mathcal{E}}(\mathbf{t}) \sim \mathcal{O}(\epsilon), \qquad \vec{\mathbf{B}} \sim \mathcal{O}(\epsilon^{\mathbf{0}})$$

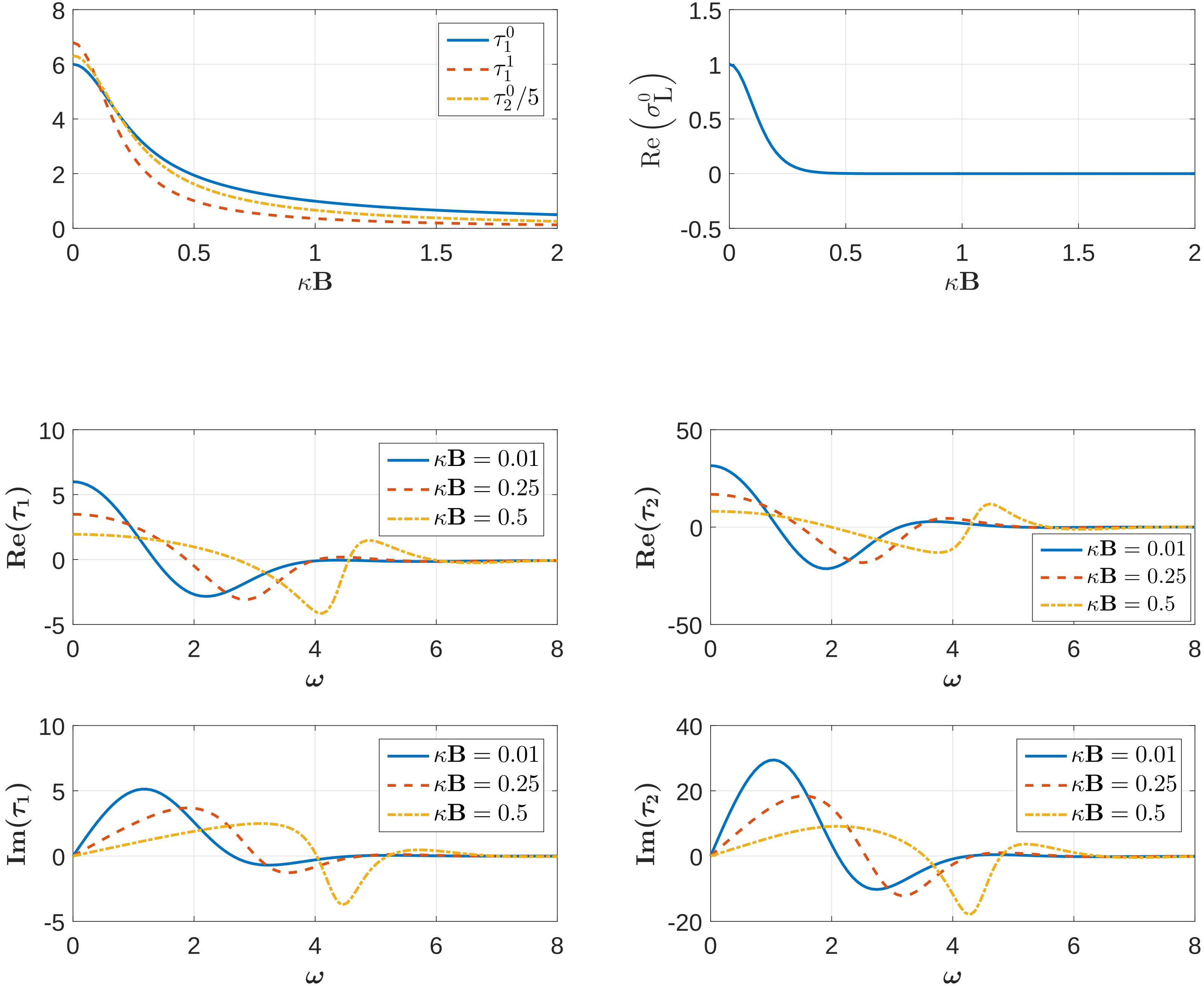
Linear in ϵ

$$\mathbf{J}^{\mathrm{t}} = \mathbf{0}, \qquad \vec{\mathbf{J}} = \sigma_{\mathrm{e}}[\partial_{\mathrm{t}}]\vec{\mathcal{E}} + \kappa\tau_{\mathbf{1}}[\partial_{\mathrm{t}}]\,\rho_{\mathbf{5}}\vec{\mathbf{B}} + \kappa^{\mathbf{2}}\tau_{\mathbf{2}}[\partial_{\mathrm{t}}]\left(\vec{\mathcal{E}}\cdot\vec{\mathbf{B}}\right)\vec{\mathbf{B}}; \qquad \mathbf{J}_{\mathbf{5}}^{\mathrm{t}} = \rho_{\mathbf{5}}, \qquad \vec{\mathbf{J}}_{\mathbf{5}} = \mathbf{0}$$

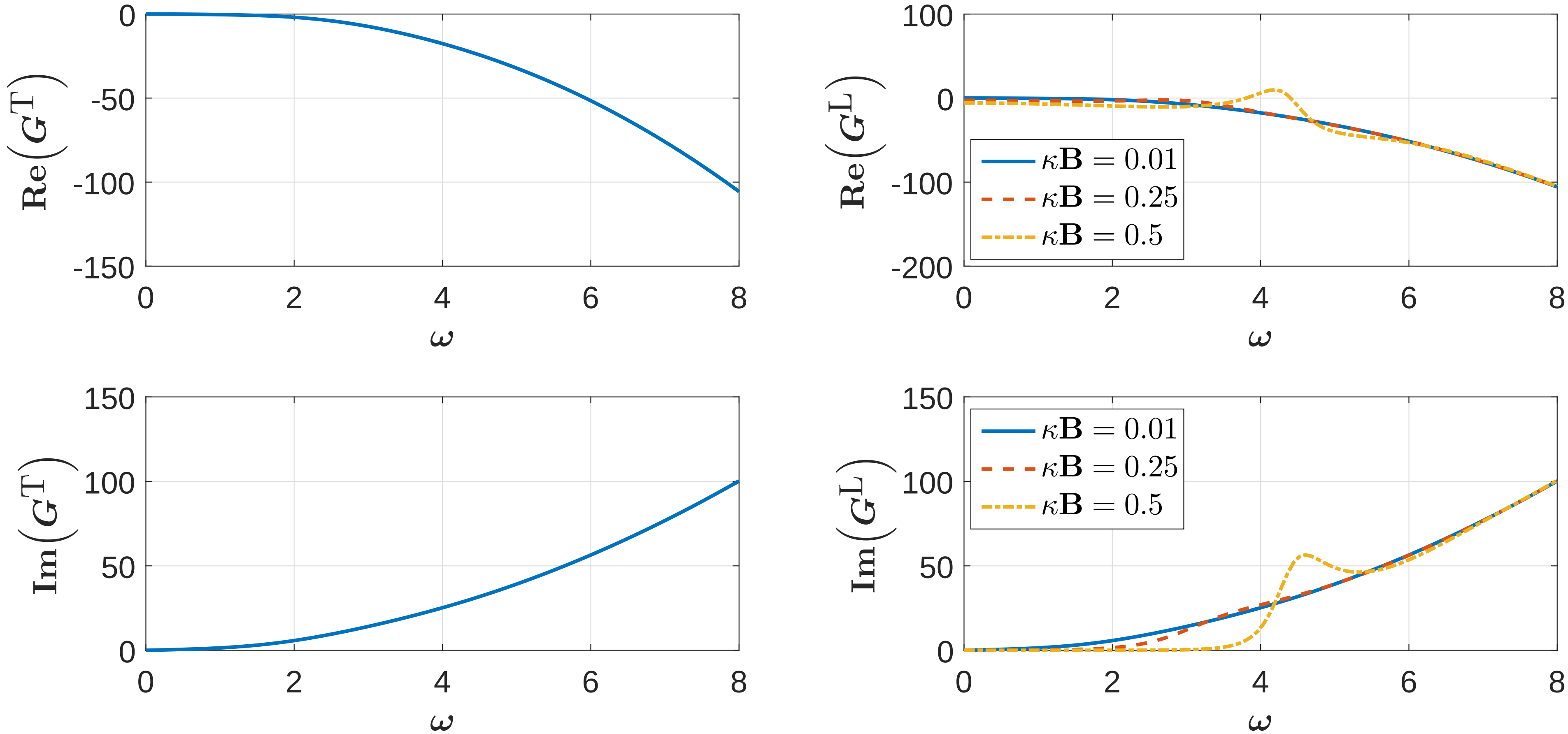
The electric current is put on-shell,

$$\mathbf{J}^{\mathrm{i}} = \sigma_{\mathrm{ij}}\mathcal{E}_{\mathrm{j}}, \qquad \sigma_{\mathrm{ij}} = \underbrace{\sigma_e}_{\sigma_{\mathbf{T}}}\left(\delta_{\mathrm{ij}} - \frac{\mathbf{B}_{\mathrm{i}}\mathbf{B}_{\mathrm{j}}}{\mathbf{B}^{\mathbf{2}}}\right) + \underbrace{\left[\sigma_e - \left(\frac{12}{i\omega}\tau_1 - \tau_2\right)\kappa^2\mathbf{B}^2\right]}_{\sigma_{\mathbf{L}}}\frac{\mathbf{B}_{\mathrm{i}}\mathbf{B}_{\mathrm{j}}}{\mathbf{B}^{\mathbf{2}}},$$

In DC limit τ_1^0 is known analytically K. Landsteiner, Y. Liu, and Y.-W. Sun, (2015); M. Ammon, S. Grieninger, A. Jimenez-Alba, R. P. Macedo, and L. Melgar, (2016)

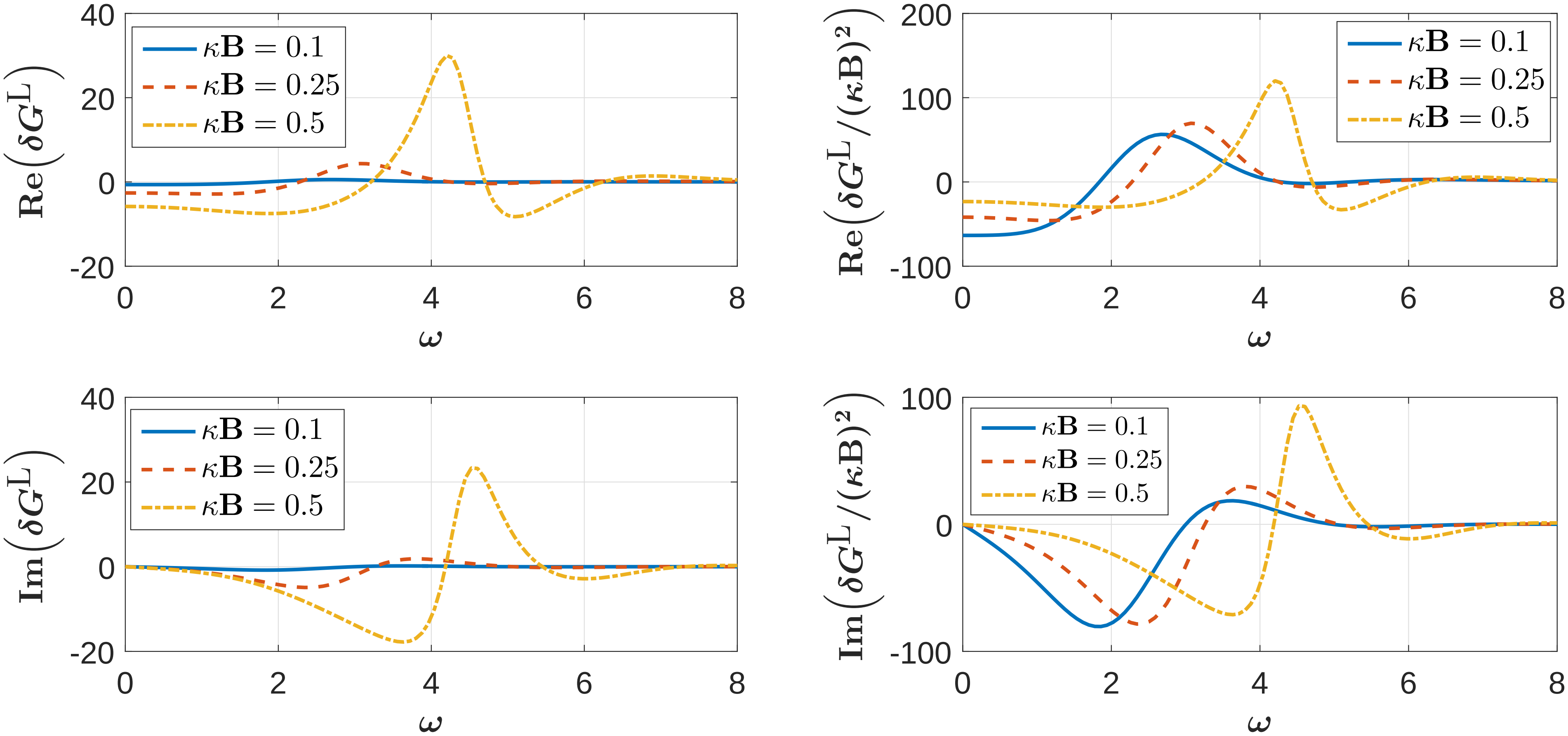


τ_1^0
τ_1^1
$\tau_2^0/5$
$\kappa\mathbf{B}$
$\mathrm{Re}\left(\sigma_{\mathrm{L}}^0\right)$
$\kappa\mathbf{B}$
$\mathbf{Re}(\tau_1)$
$\kappa\mathbf{B} = 0.01$
$\kappa\mathbf{B} = 0.25$
$\kappa\mathbf{B} = 0.5$
ω
$\mathbf{Re}(\tau_2)$
$\kappa\mathbf{B} = 0.01$
$\kappa\mathbf{B} = 0.25$
$\kappa\mathbf{B} = 0.5$
ω
$\mathbf{Im}(\tau_1)$
$\kappa\mathbf{B} = 0.01$
$\kappa\mathbf{B} = 0.25$
$\kappa\mathbf{B} = 0.5$
ω
$\mathbf{Im}(\tau_2)$
$\kappa\mathbf{B} = 0.01$
$\kappa\mathbf{B} = 0.25$
$\kappa\mathbf{B} = 0.5$
ω



$\mathbf{Re}\left(G^{\mathrm{T}}\right)$
ω
$\mathbf{Re}\left(G^{\mathrm{L}}\right)$
$\kappa\mathbf{B} = 0.01$
$\kappa\mathbf{B} = 0.25$
$\kappa\mathbf{B} = 0.5$
ω
$\mathbf{Im}\left(G^{\mathrm{T}}\right)$
ω
$\mathbf{Im}\left(G^{\mathrm{L}}\right)$
$\kappa\mathbf{B} = 0.01$
$\kappa\mathbf{B} = 0.25$
$\kappa\mathbf{B} = 0.5$
ω

To enhance the effect of anomaly, $\delta G^{\mathbf{L}} = G^{\mathbf{L}} - G^{\mathbf{T}}$



Conclusions

- An off-shell constitutive relation for $U(1)$ current consists of a momenta-dependent diffusion term and two conductivities. Certain universality between dissipative transport coefficients η and $\mathcal{D}$ is observed.

- Causality restoration: at large momenta, the effective diffusion TCF is a decreasing function of both frequency; the corresponding memory function has support in the past only.

- We have re-examined transport coefficients induced by the chiral anomaly. We seem to be able to rediscover all known anomaly-induced effects within one and the same holographic model, without introducing any additional inputs or model assumptions, which appeared in the literature

- For linearized problem, we have completely determined all anomaly-induced TCFs to all orders in the gradient expansion

- For nonlinear problem with constant external fields, we have found E-induced corrections to CME/CSE, and B-induced modifications to CMW, and diffusion coefficient D

- There seem to be enhancements of anomaly-induced affects at finite frequencies